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A RATIONAL ROUTE TO RANDOMNESS

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The concept of *adaptively rational equilibrium* (A.R.E.) is introduced. Agents adapt their beliefs over time by choosing from a finite set of different predictor or expectations functions. Each predictor is a function of past observations and has a performance or fitness measure which is publicly available. Agents make a rational choice concerning the predictors based upon their past performance. This results in a dynamics across predictor choice which is coupled to the equilibrium dynamics of the endogenous variables.

As a simple, but typical, example we consider a cobweb type demand-supply model where agents can choose between rational and naive expectations. In an unstable market with (small) positive information costs for rational expectations, a high intensity of choice to switch predictors leads to highly irregular equilibrium prices converging to a strange attractor. The irregularity of the equilibrium time paths is explained by the existence of a so-called homoclinic orbit and its associated complicated dynamical phenomena. Thus local instability and global complicated dynamics may be a feature of a fully rational notion of equilibrium.

KEYWORDS: Heterogeneous beliefs, adaptive learning, business cycles, cobweb model, homoclinic bifurcations, and strange attractors.

1. INTRODUCTION

THIS PAPER PUTS FORTH a concept of *adaptively rational equilibrium* (A.R.E.) where agents base decisions upon predictions of future values of endogenous variables whose actual values are determined by equilibrium equations. Predictions are made by choosing from a finite set P of predictor or expectations functions. These predictors are functions of past information. Each predictor has a performance measure attached to it which is publicly available to all agents. Agents use a discrete choice model along the lines of Manski and McFadden (1981) and Anderson, de Palma, and Thisse (1993) to pick a predictor from P where the deterministic part of the utility of the predictor is the

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performance measure. This results in the *adaptive rational equilibrium dynamics* (*A.R.E.D.*), a dynamics across predictor choice which is coupled to the equilibrium dynamics of the endogenous variables.

The A.R.E.D. incorporates a general mechanism, which can generate local instability of the steady state and complicated global equilibrium dynamics. Let us explain this general mechanism by discussing the simple case with two predictors. Agents can either buy at small but positive information costs C a sophisticated predictor H_1 (e.g., rational expectations or long memory predictors such as "Ljung-type" predictors used by Marcet and Sargent (1989) or Evans and Honkapohja (1994)) or freely obtain another simple predictor H_2 (e.g., adaptive, short memory, or naive expectations). Agents make a rational decision between predictors and tend to choose the predictor which yields the smallest prediction error or the highest net profit. Suppose that if all agents would use the sophisticated predictor H_1, all time paths of the endogenous variables, say prices, would converge to a unique *stable* steady state, whereas when all agents would use the simple predictor H_2, the same unique steady state would occur, but this time it would be *unstable*. Consider an initial state with prices close to the steady state value and almost all agents using the simple predictor. Then prices will diverge from their steady state value and the prediction error from predictor H_2 will increase. As a result, the number of agents who are willing to pay some information costs to get the predictor H_1 increases. When the intensity of choice to switch between the two beliefs is high, as soon as the net profit associated to predictor H_1 is higher than the net profit associated to H_2, almost all agents will switch to H_1. Prices are then pushed back towards their steady state value and remain there for a while. With prices close to their steady state value, the prediction error of predictor H_2 becomes small again whereas net profit corresponding to predictor H_1 becomes negative because of the information costs. When the intensity of choice is high, most agents will switch their beliefs to predictor H_2 again, and the story repeats. There is thus one "centripetal force" of "far-from-equilibrium" negative feedback when most agents use the sophisticated predictor and another "centrifugal force" of "near-equilibrium" positive feedback when all agents use the simple predictor. The interaction between these two opposing forces can lead to very complicated Adaptive Rational Equilibrium Dynamics when the intensity of choice to switch beliefs is high. In an unstable market with information costs for sophisticated predictors, local instability and irregular dynamics may thus be a feature of a fully rational notion of equilibrium.

The paper aims at making this intuitive description rigorous. In fact the primary contribution of the paper is to show that in binary predictor choice settings whose dynamics are reducible to two dimensional discrete systems, a rational choice between a cheap destabilizing predictor and a costly stabilizing one, leads to the existence of a near homoclinic orbit when the intensity of choice to switch predictors is high. As already pointed out by Poincaré (1890) a century ago, existence of homoclinic orbits implies very complicated dynamics. Applying recent mathematical results from homoclinic bifurcation theory (see

Palis and Takens (1993) for an excellent treatment) we prove existence of strange attractors in a simple two-dimensional binary predictor system.

In the literature on chaos in economic models, most examples are in fact given by a one-dimensional difference equation $x_{t+1} = f(x_t)$, where f is a non-monotonic map. One of the most prominent examples is the overlapping generations (OLG) model (Benhabib and Day (1982) and Grandmont (1985)). Our models with heterogeneous beliefs yield discrete dynamic models of at least two state variables, say prices and fractions of the different groups. The homoclinic bifurcation theory which we use can be applied to other two-dimensional nonlinear economic models as well. In two related and stimulating papers, de Vilder (1995, 1996) recently introduced homoclinic bifurcation theory into economics, presenting "computer assisted proofs" of the existence of homoclinic orbits and the associated complicated dynamic phenomena in a two-dimensional OLG model with a Leontief production technology. Occurrence of homoclinic bifurcations was shown by developing the DUNRO computer program (Sands and de Vilder (1994)) for accurate numerical computation of stable and unstable manifolds of periodic saddles in general two-dimensional systems. Recently, Pintus, Sands, and de Vilder (1996) used the DUNRO program to show complicated equilibrium dynamics in an OLG economy with a CES production technology.

In the present paper, we focus on a simple two-dimensional example of an A.R.E.D., namely the cobweb model with rational versus naive expectations and linear demand and supply. This simple example displays many of the essential features that have been observed for a fairly general class of binary choice predictive systems. Other examples can be found in the longer working paper version of this paper (Brock and Hommes (1995)). Numerical simulations indicate that the conflict between the stabilizing and destabilizing forces leads to complicated dynamics in higher-dimensional predictor choice systems as well.

A secondary contribution of the paper is the formulation of two general local instability theorems which show that, in higher-dimensional cobweb type predictive systems, the tension between stabilizing and destabilizing forces causes the dynamics almost surely *not* to settle down to the equilibrium steady state. However, in higher-dimensional cases the exact nature of the global dynamics around the unstable steady state remains unknown.

At this point we would like to refer to some other related work. Arthur (1992) and Arthur, Holland, LeBaron, Palmer, and Taylor (1994) have written interesting papers on "artificial economic life" where agents use a mix of predictors to adapt to an economic environment. De Grauwe, DeWachter, and Embrechts (1993) consider models of exchange rates with two classes of agents, chartists and fundamentalists, and Chiarella (1992), Lux (1995), and Sethi (1996) consider simple continuous-time stock market models with chartists and fundamentalists. One can view the present paper as providing some analytic results for these more numerically oriented papers.

The paper is organized as follows. Section 2 describes the general A.R.E.D. with K predictors and presents two general local-instability-of-the-steady-state theorems. The main section is Section 3, analyzing the global A.R.E.D. in the cobweb model with rational versus naive expectations. Section 4 concludes, and an Appendix contains the proofs of the results.

2. ADAPTIVE RATIONAL EQUILIBRIUM DYNAMICS (A.R.E.D.)

Adaptive rational equilibrium dynamics is a coupling between market equilibrium dynamics and predictor selection. In each period, agents make a rational choice among a finite set of predictors, based upon their past performance. We concentrate on the familiar cobweb demand-supply model, but note that the A.R.E.D. concept can be incorporated into a more general temporary equilibrium model (e.g., Grandmont and Laroque (1991), Grandmont (1994)). Brock and Hommes (1996a, 1996b, 1996c) investigate the A.R.E.D. in the present discounted value asset pricing model, and de Fontnouvelle (1995) in a financial market model with informed and uniformed traders. Brock and LeBaron (1996) consider a structural stochastic financial model with adaptive beliefs.

The cobweb model describes fluctuations of equilibrium prices in a market of a nonstorable good that takes one time unit to produce. Let $D(p_t)$ be the demand and $S(p_t^e)$ be the supply of the good, where p_t is actual price and p_t^e producers expected price, made at the beginning of period t. Supply is derived from firms maximizing profits with a cost function $c(q)$, so

$$S(p_{t+1}^e) = \operatorname{argmax}\{p^e q - c(q)\} = (c')^{-1}(p_{t+1}^e). \tag{2.1}$$

In the case of *homogeneous beliefs* all producers use the same expectations or predictor function $p_{t+1}^e = H(\vec{P}_t)$, where $\vec{P}_t = (p_t, p_{t-1}, \ldots, p_{t-L})$ is a vector of past prices. Equilibrium price dynamics is then described by

$$D(p_{t+1}) = S\left(H\left(\vec{P}_t\right)\right) \quad \text{or} \quad p_{t+1} = D^{-1}S\left(H\left(\vec{P}_t\right)\right), \tag{2.2}$$

with D^{-1} the inverse demand function. We assume that demand D is decreasing and supply S is increasing, so (2.2) is well defined.[2]

In this paper we consider the equilibrium price dynamics in the cobweb model with *heterogeneous beliefs*, where agents can choose between K different predictors $H_1, H_2, \ldots, H_K$. The fractions $n_{j,t}$, $1 \le j \le K$, of agents using predictor H_j in period t, will change over time. In the cobweb model with heterogeneous beliefs market equilibrium is determined by

$$D(p_{t+1}) = \sum_{j=1}^{K} n_{j,t}\left(p_t, \mathscr{H}\left(\vec{P}_{t-1}\right)\right) S\left(H_j\left(\vec{P}_t\right)\right) \tag{2.3}$$

where as before $\vec{P}_t = (p_t, p_{t-1}, \ldots, p_{t-L})$ is a vector of past prices and $\mathscr{H}(\vec{P}_{t-1})$ is the vector of predictions $(H_1(\vec{P}_{t-1}), \ldots, H_K(\vec{P}_{t-1}))$.

[2] The price dynamics in the cobweb model with homogeneous beliefs (2.2) and nonlinear, but *monotonic*, demand and supply curves, has recently been analyzed for various expectations schemes in Hommes (1991, 1994, 1997). In the case of homogeneous beliefs several bifurcation routes to chaos can arise.

We now describe how the fractions $n_{j,t}$ evolve over time. Predictor choice is based upon their past performance. Here we choose past realized net profits π_j as the publicly available performance or fitness measure U_j, but other choices (e.g., past squared prediction errors) yield similar results. Realized net profits $\pi_j(p_{t+1}, H_j(\vec{P}_t))$, $1 \leq j \leq K$, from using predictor $H_j(\vec{P}_t)$ when the actual equilibrium price becomes p_{t+1} equals

$$\pi_j\left(p_{t+1}, H_j\left(\vec{P}_t\right)\right) = p_{t+1} S\left(H_j\left(\vec{P}_t\right)\right) - c\left(S\left(\vec{H}_j(P_t)\right)\right) - C_j \tag{2.4}$$

where $c(\cdot)$ is the producer's cost function and $C_j \geq 0$ are information costs for obtaining predictor H_j. For a simple "habitual rule of thumb" predictor (e.g., naive or adaptive expectations) information costs $C_j = 0$, but for a more "sophisticated" predictor (e.g., rational expectations, fundamentalist beliefs, or OLS-learning) information costs C_j may be positive. The performance measure U_j for predictor H_j is a weighted average of past realized net profits:

$$U_{j,t+1} \equiv \sum_{k=0}^{M} w_{j,k} \pi_j\left(p_{t+1-k}, H_j\left(\vec{P}_{t-k}\right)\right), \qquad \sum_{k=0}^{M} w_{j,k} = 1. \tag{2.5}$$

The updated fraction $n_{j,t+1}$ of firms using predictor H_j in the next period is

$$n_{j,t+1} \equiv n_{j,t+1}\left(p_{t+1}, \mathscr{H}\left(\vec{P}_t\right)\right) = \exp[\beta U_{j,t+1}]/Z_{t+1}, \tag{2.6}$$

$$Z_{t+1} \equiv \sum_{j=1}^{K} \exp[\beta U_{j,t+1}],$$

where Z_{t+1} is just a normalization so that the fractions $n_{j,t+1}$ add up to 1. The *cobweb model with heterogeneous beliefs* is thus defined by (2.3) and (2.6). The timing of predictor selection is important. After the equilibrium price p_{t+1} has been revealed by (2.3), all predictors H_j are evaluated according to their publicly available performance measure $U_{j,t+1}$ and the new fractions are determined by (2.6). These updated fractions then determine the next equilibrium price p_{t+2} by (2.3), etc. It will be clear from (2.6) that most agents will choose predictors that yielded higher net profits over the past. The parameter β is the *intensity of choice* measuring how fast agents switch predictors. The special limiting case $\beta = +\infty$ corresponds to the *neoclassical deterministic choice model*, where in each period all agents choose the optimal predictor. We will particularly be interested in the equilibrium dynamics for high but finite values of the intensity of choice β.

The fractions (2.6) are exactly the limiting fractions or probabilities of choosing H_j in a stochastic *discrete choice model* for predictor selection. Discrete choice models give analytically tractible choice probabilities for which highly developed theory exists in both econometrics and statistical physics. See Manski and McFadden (1981) or Anderson, de Palma, and Thisse (1993) for an extensive treatment and background information on discrete choice modelling.

Discrete choice setups also easily generalize to models with social interactions as shown by Brock (1993). See also Kiefer, Ye, and An (1993).[3]

We shall call the dynamical system given by (2.3) and (2.6) the *adaptive rational equilibrium dynamics* (*A.R.E.D.*). This name is intended to capture two key features: (i) Predictor choice is itself a purposive economic act based upon a rational decision which itself should be modelled and should be part of the equilibrium concept studied; and (ii) predictor choice feeds into the market equilibrium dynamics, which in turn feeds into predictor choice.

These ideas could be viewed as a variation and extension on the work of Evans and Ramey (1992) on "Calculation Equilibrium" as well as Arthur (1992), Arthur et al. (1994), Blume and Easley (1992), Cabrales and Hoshi (1996), Kurz (1994), Marimon (1993), Sargent (1993), and Townsend (1983). Our interest here is on the extra dynamical system structure that is added by the interaction of purposive dynamical choice across predictors with the conventional market equilibrium dynamics.

We next formulate two general local instability theorems for the cobweb type A.R.E.D. (2.3)–(2.6). Let p^* be the price corresponding to the intersection point of demand and supply, i.e. $D(p^*) = S(p^*)$, and let $\vec{P}^*$ denote the L-dimensional vector $(p^*, \ldots, p^*)$. We make the following assumption:

ASSUMPTION A1: *$D(\cdot)$ is strictly decreasing and positive, $S(\cdot)$ is strictly increasing and positive except at zero, and $S(0) = 0$. The functions D, S, $\{H_j\}$ are all C^1, i.e., continuously differentiable. For each predictor H_j, the dynamical system $p_{t+1} = D^{-1}[S(H_j(\vec{P}_t))]$ has p^* as its unique steady state.*

According to Assumption A1, the cobweb model with homogeneous beliefs H_j has p^* as the unique steady state equilibrium. It is then easy to show that p^* is also an equilibrium steady state of the A.R.E.D. (2.3)–(2.6).[4] Consider

$$D(p_t) = \frac{1}{K} \sum_{j=1}^{K} S\left(H_j\left(\vec{P}_{t-1}\right)\right), \tag{2.7}$$

the cobweb model with heterogeneous beliefs, with agents uniformly distributed over the predictor space and all fractions of agents fixed at $1/K$. We assume the following:

ASSUMPTION A2: *The steady state equilibrium p^* or $\vec{P}^* \equiv (p^*, \ldots, p^*)$ of (2.7), is hyperbolic (i.e. the linearization of (2.7) at p^* has no eigenvalues on the unit circle), and (2.7) is locally unstable at p^*.*

[3] These systems can also generate thresholding behavior and endogenous regime changes. These features relate naturally to work in macroeconomics such as Azariadis and Smith (1994). A nice survey on complex dynamics and endogenous fluctuations in macroeconomics is Guesnerie and Woodford (1992).

[4] Assumption A1 excludes, e.g. constant, biased predictors $H_j(\vec{P}_{t-1}) = \bar{p}$. Different biased predictors imply that agents "believe" in different steady states. This introduces an additional source of instability in the A.R.E.D., not considered in this paper; but see Brock and Hommes (1996b) on the role of biased predictors in asset pricing models.

This assumption means that when all fractions of agents using predictor H_j are fixed at $1/K$, the corresponding equilibrium dynamics (2.7) has an unstable steady state p^*. According to the following theorem, the steady state of the A.R.E.D. with switching of beliefs is then also locally unstable.

THEOREM 2.1: *Assume* $A1$, $A2$ *and information costs* $C_j = 0$ *for all* H_j. *Then the* $A.R.E.D.$ (2.3)–(2.6) *has a locally unstable steady state.*

Proofs of all results are given in the Appendix. The theorem says that an unstable equilibrium steady state with heterogeneous agents equally distributed over the space of predictors, cannot be stabilized by switching of beliefs. A second local instability result arises when there are *information costs*. Let $C_j \geq 0$ denote the costs for agents to obtain predictor H_j, and assume $C_1 > C_2 > \cdots > C_K \geq 0$. We continue to assume A1 but replace A2 by the following assumption.

ASSUMPTION A2′: *When all agents use the cheapest predictor* H_K, *the steady state* $\vec{P}^* \equiv (p^*, \ldots, p^*)$ *is hyperbolic and locally unstable* (*i.e. the simple cobweb model* (2.2) *with homogeneous beliefs* H_K *is locally unstable at* p^*).

The second local instability theorem is as follows.

THEOREM 2.2: *Assume* A_1, A_2' *and* $C_1 > C_{K-1} > C_K \geq 0$. *When the intensity of choice* β *is sufficiently large, then the steady state* $\vec{P}^* \equiv (p^*, \ldots, p^*)$ *of the* $A.R.E.D.$ (2.3)–(2.6) *is locally unstable.*

For large β, at the equilibrium steady state p^*, most agents will use the cheapest predictor H_K. In economic models, it is natural to take the cheapest predictor to be some myopic, naive, or habitual "rule of thumb." It is not difficult to imagine markets that are unstable under such prediction rules. Our theorem says that local instability is inherent in such situations when more complicated or sophisticated prediction methods are more expensive.

3. THE COBWEB MODEL WITH RATIONAL VERSUS NAIVE EXPECTATIONS

In this section we investigate the global equilibrium dynamics in a simple, example of an A.R.E.D.: the cobweb model with rational versus naive expectations and linear demand and supply. Although this example is simple, it captures many of the essential features occurring in other cases, such as fundamentalists' beliefs versus naive expectations or long memory, "Ljung"-type predictors versus naive, analyzed in Brock and Hommes (1995). The section is subdivided into five subsections. The first presents the model. The second investigates the stability of the steady state and existence of a period 2 cycle. Subsection 3.3 briefly discusses Poincaré's notion of homoclinic orbits and some related recent mathematical results on homoclinic bifurcations. In Subsection 3.4 we investigate the global dynamics of the model and in particular we show the existence of strange attractors when the intensity of choice to switch predictors is high. Finally, in Subsection 3.5 we focus on coexistence of stable cycles as a source of complicated equilibrium dynamics and use the DUNRO program (Sands and de Vilder (1994)) to detect the primary heteroclinic and homoclinic bifurcations as the intensity of choice increases.

3.1. *The Model*

Market equilibrium in the cobweb model with two predictors is determined by

$$D(p_{t+1}) = n_{1,t} S\left(H_1\left(\vec{P}_t\right)\right) + n_{2,t} S\left(H_2\left(\vec{P}_t\right)\right) \tag{3.1}$$

where $n_{1,t}$ and $n_{2,t}$ are the fractions of agents using predictors H_1, respectively H_2. To keep the model as simple as possible, we choose a linear demand curve and a linear supply curve (derived from a quadratic cost function $c(q) = q^2/2b$):

$$D(p_t) = A - Bp_t, \tag{3.2a}$$

$$S\left(H\left(\vec{P}_t\right)\right) = bH\left(\vec{P}_t\right). \tag{3.2b}$$

Throughout this section H_1 will be *rational expectations* (perfect foresight),[5] whereas H_2 will be *naive expectations*, so

$$H_1\left(\vec{P}_t\right) = p_{t+1}, \tag{3.3a}$$

$$H_2\left(\vec{P}_t\right) = p_t. \tag{3.3b}$$

Naive expectations are freely available, whereas rational expectations can be obtained at information costs $C \geq 0$, or equivalently, the price difference between the two predictors is $C \geq 0$. Information costs represent an extra effort that agents must face in obtaining a more sophisticated price forecast such as rational expectations. To keep the global dynamics of the model analytically tractable, we choose net realized profits in the last period as the performance measure for predictor selection.[6] Using linear demand and supply (3.2), rational versus naive expectations, (3.3), and realized net profits in the last period, the general performance measure (2.4) reduces to

$$\begin{aligned} \pi_1(p_{t+1}, p_{t+1}) &= \frac{b}{2} p_{t+1}^2 - C, \\ \pi_2(p_{t+1}, p_t) &= \frac{b}{2} p_t (2p_{t+1} - p_t), \end{aligned} \tag{3.4}$$

[5] Strictly speaking our notation of H_1 in (3.3a) is not correct, since the right-hand side is not a function of the vector $\vec{P}_t$ of past prices.

[6] A weighted average of past net realized profits would perhaps be more realistic. Choosing only one lag in the performance measure leads to a two-dimensional nonlinear difference equation, which is analytically tractable. Introducing more lags leads to higher-dimensional nonlinear systems, for which few global mathematical results are available. Numerical simulations indicate, however, that more lags yield similar results.

where C is information costs for rational expectations. After observing the equilibrium price p_{t+1}, the updated fractions of agents using rational, respectively naive, expectations in the next period are (cf. (2.6))

$$n_{1,t+1} = \exp\left[\beta\left(\frac{b}{2}p_{t+1}^2 - C\right)\right]/Z_{t+1}, \tag{3.5a}$$

$$n_{2,t+1} = \exp\left[\beta\left(\frac{b}{2}p_t(2p_{t+1} - p_t)\right)\right]/Z_{t+1}, \tag{3.5b}$$

where Z_{t+1} is the sum of the numerators. As before, the parameter β is the *intensity of choice*, measuring how fast agents switch predictors. It will be convenient to introduce the difference between the two fractions, i.e. $m_{t+1} = n_{1,t+1} - n_{2,t+1}$, so $m_{t+1} = 1$ ($m_{t+1} = -1$) corresponds to all agents using predictor H_1 (H_2). Using (3.5) we obtain

$$m_{t+1} = \mathrm{Tanh}\left(\frac{\beta}{2}\left[\frac{b}{2}(p_{t+1} - p_t)^2 - C\right]\right). \tag{3.6}$$

Using $n_{1,t} = (m_t + 1)/2$, $n_{2,t} = (1 - m_t)/2$, linear demand and supply in (3.2) and rational and naive expectations in (3.3), market equilibrium in (3.1) becomes

$$p_{t+1} = \left(A - \frac{1+m_t}{2}bp_{t+1} - \frac{1-m_t}{2}bp_t\right)/B. \tag{3.7}$$

Note that agents employing rational expectations take into account that there are two classes of agents with different beliefs, and thus they are assumed to have perfect knowledge about market equilibrium equations, prices, and also about the fractions of all belief types. Without loss of generality we change coordinates and choose the steady state price p^*, the intersection of demand and supply, as the new origin, so that p_t represents (positive or negative) deviations from the steady state.[7] For our linear supply and demand curves this simply means fixing $A = 0$ in (3.7). Solving for p_{t+1} then yields

$$p_{t+1} = \frac{-b(1-m_t)p_t}{2B + b(1+m_t)} = f(p_t, m_t) \tag{3.8}$$

and substituting (3.8) into (3.6) yields

$$m_{t+1} = \mathrm{Tanh}\left(\frac{\beta}{2}\left[\frac{b}{2}\left(\frac{b(1-m_t)}{2B + b(1+m_t)} + 1\right)^2 p_t^2 - C\right]\right) = g(p_t, m_t). \tag{3.9}$$

[7] From (3.6) it is immediately clear that changing the origin also does not affect predictor choice; measuring in deviations from steady state gives exactly the same expression for m_{t+1}.

The adaptive rational equilibrium dynamics (A.R.E.D.) of the cobweb model with rational versus naive expectations and linear demand and supply is thus described by the two-dimensional system of *nonlinear* difference equations (3.8)–(3.9). Write F_β for the corresponding two-dimensional map. We will investigate the dynamics of F_β as the intensity of choice β increases.

3.2. *Stability of the Steady State and Existence of Period 2 Cycle*

A simple computation shows that the model (3.8)–(3.9) has a unique steady state $E = (0, \overline{m}(\beta)) = (0, \text{Tanh}(-\beta C/2))$. The stability results of the steady state are as follows.

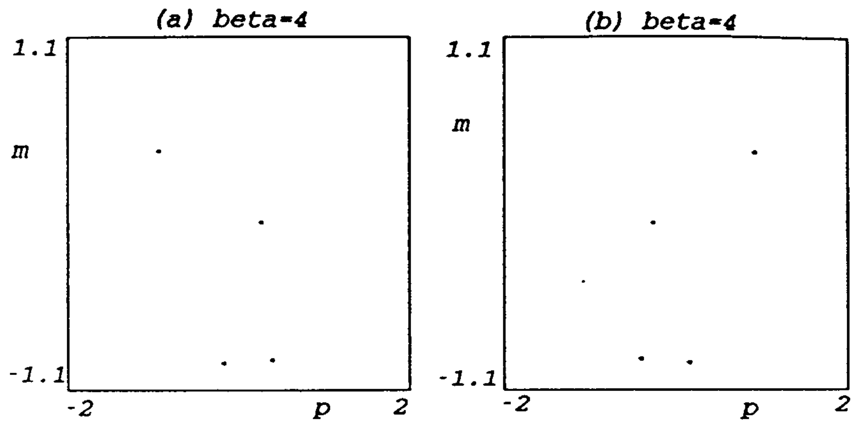
THEOREM 3.1: *Assume that the slopes of supply and demand satisfy* $b/B > 1$.

(i) *When the information costs* $C = 0$, *the steady state* $E = (0,0)$ *and it is always globally stable.*

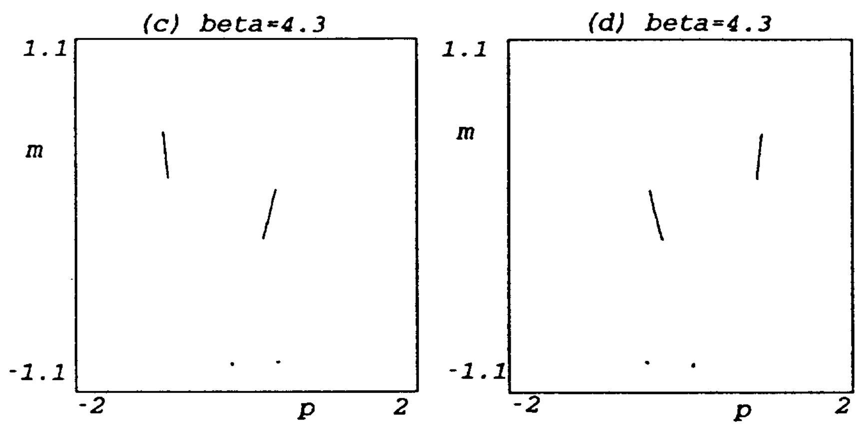
(ii) *When the information costs* $C > 0$, *then there exists a critical value* β_1 *such that for* $0 \leq \beta < \beta_1$ *the equilibrium is globally stable, while for* $\beta > \beta_1$ *the equilibrium is an unstable saddle point with eigenvalues* 0 *and* $\lambda(\beta) = -b(1-\overline{m}(\beta))/[2B + b(1+\overline{m}(\beta))] < -1$. *At the critical value* β_1 *the steady state value* $\overline{m}(\beta_1) = -B/b$.

(iii) *When the steady state is unstable, there exists a locally unique period 2 orbit* $\{(\bar{p}, \tilde{m}), (-\bar{p}, \tilde{m})\}$, *with* $\tilde{m} = -B/b$ *and* $\bar{p}$ *the unique positive solution of* $\text{Tanh}[(\beta/2)(2b\bar{p}^2 - C)] = -B/b$. *There exists a* $\beta_2 > \beta_1$ *such that the period 2 cycle is stable for* $\beta_1 < \beta < \beta_2$.

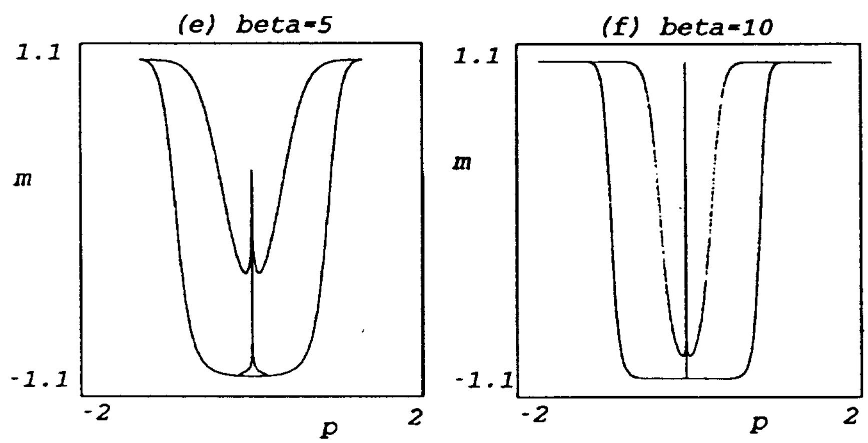
The assumption $(b/B) > 1$ means that if all agents employ naive expectations, the market will be unstable. In the A.R.E.D., when there are no information costs for rational expectations the steady state is globally stable. However, when information costs are positive, no matter how small, a large intensity of choice to switch beliefs yields an unstable saddle point equilibrium steady state and the creation of a two-cycle. As the intensity of choice increases, the *primary bifurcation* is thus a period doubling bifurcation, in which the steady state loses stability and a stable two-cycle is created. What happens to this stable two-cycle when the intensity of choice increases? Figure 1 shows some typical attractors in the phase space for increasing values of β. The two-cycle loses stability through a secondary bifurcation in which two different *coexisting* stable four-cycles are created (Figures 1a–b). Notice that the two stable four-cycles are symmetric with respect to the m-axis $p = 0$, due to the fact that the map f in (3.8) is odd, i.e. $f(-p, m) = -f(p, m)$. In Subsection 3.5 we will investigate the secondary bifurcation and the coexistence of attractors in more detail. As β further increases, apparently both stable four-cycles turn into four-piece chaotic attractors (Figure 1c–d) after a cascade of infinitely many period doubling bifurcations. The two coexisting four-piece chaotic attractors are also symmetric with respect to the m-axis. As β is further increased, our numerical simulations



(a)-(b) Two coexisting stable 4-cycles.



(c)-(d) Two coexisting 4-piece chaotic attractors.



(e)-(f) Chaotic attractors.

FIGURE 1.—Attractors for different β-values with $A = 0$, $B = 0.5$, $b = 1.35$, and $C = 1$.

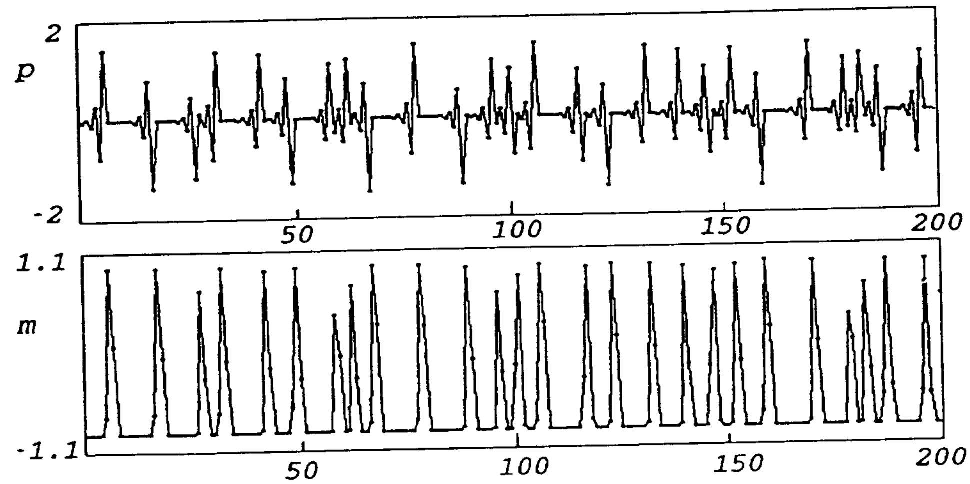


FIGURE 2.—Time series of prices p_t and the difference in fractions m_t for $\beta = 5$, corresponding to the chaotic attractor in Figure 1e.

indicate the occurrence of a chaotic attractor (Figure 1e–f).[8] Time series of equilibrium prices and differences of fractions are shown in Figure 2.

3.3. *Homoclinic Orbits and Complicated Dynamic Phenomena*

The occurrence of erratic equilibrium time paths, for large values of the intensity of choice β, turns out to be closely related to the existence of so-called *homoclinic points*. This notion was introduced already by Poincaré (1890) at the turn of the century. A homoclinic point $q \neq p$ is an intersection point between the stable and unstable manifolds of a steady state (or periodic) saddle point p. Obviously, a linear system cannot have homoclinic points; however, nonlinear systems often have homoclinic points. Poincaré observed that the existence of a homoclinic point implies very complicated dynamical behavior. Only in the last decades, however, has a much better mathematical understanding of the dynamic consequences associated to homoclinic orbits emerged. In particular, recently it has been shown that existence of strange, chaotic attractors is closely related to homoclinic behavior. Since most economists are not familiar with these notions we recall Poincaré's homoclinic orbits and briefly discuss some recent mathematical results concerning complicated dynamical behavior in

[8] A strange or chaotic attractor is an attractor that is the closure of an unstable manifold of some periodic saddle point and contains a dense orbit with positive Lyapunov exponent (e.g. Palis and Takens (1993, pp. 138–143)). The strange attractors in Figures 1e and 1f almost look like one-dimensional curves. Apparently the fractal dimension is close to 1 because of the strong contraction (the steady state has an eigenvalue 0) in the stable direction.

two-dimensional discrete systems. For an extensive mathematical treatment we strongly recommend the excellent book by Palis and Takens (1993), which also contains references to the original papers. A nice nontechnical introduction has recently been given by de Vilder (1995, 1996), who also presents computer assisted proofs of the existence of homoclinic orbits and related dynamic phenomena in a two-dimensional OLG model with production.

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth (i.e. differentiable) map and $x_{t+1} = F(x_t)$ the corresponding dynamical system. Let p be a *saddle* fixed point of F. The Jacobian $JF(p)$ has two real eigenvalues λ and μ such that $|\lambda| < 1$ and $|\mu| > 1$. The *local stable* and *unstable manifolds* of p are defined as

$$\text{(3.10a)} \quad W^s_{loc}(p) = \left\{x \in U \mid \lim_{n \to \infty} F^n(x) = p\right\},$$

$$\text{(3.10b)} \quad W^u_{loc}(p) = \left\{x \in U \mid \lim_{n \to -\infty} F^n(x) = p\right\},$$

where U is some (small) neighborhood of p. According to the local stable and unstable manifold theorems $W^s_{loc}(p)$ and $W^u_{loc}(p)$ are smooth curves tangent to the stable and unstable eigenspaces of $JF(p)$. The *global stable* and *unstable manifolds* are then defined as

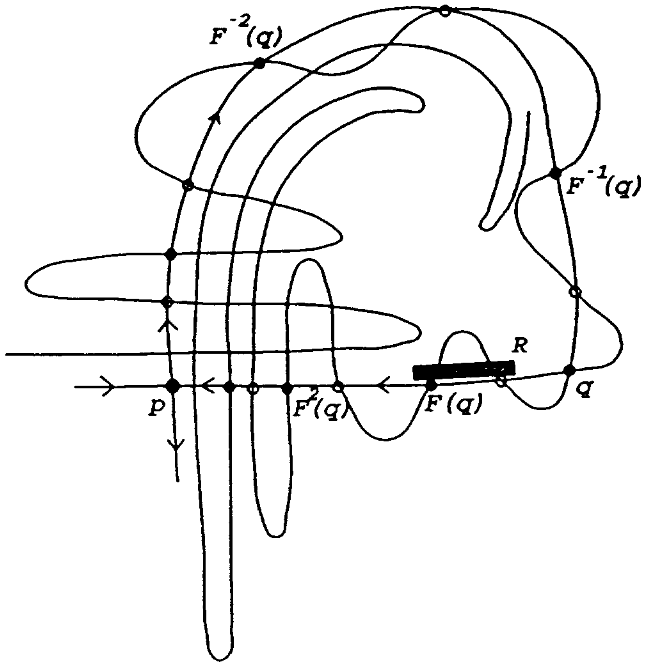
$$\text{(3.11a)} \quad W^s(p) = \bigcup_{n=0}^{\infty} F^{-n}(W^s_{loc}(p)),$$

$$\text{(3.11b)} \quad W^u(p) = \bigcup_{n=0}^{\infty} F^{n}(W^u_{loc}(p)).$$

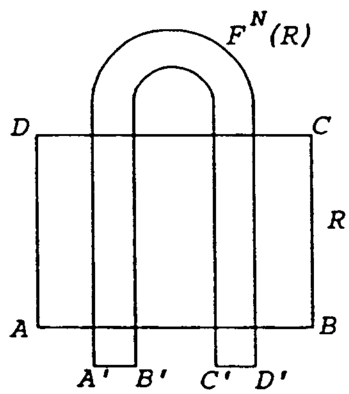
If F is a diffeomorphism (a smooth map with a smooth inverse) the stable and unstable manifolds are curves without self-intersections. If F is noninvertible, $W^u(p)$ can have self-intersections and $W^s(p)$ can have several components.

A *homoclinic point* $q \neq p$ is an intersection point of the stable and unstable manifolds $W^s(p)$ and $W^u(p)$; if $W^s(p)$ and $W^u(p)$ are tangent at q we say that q is a point of *homoclinic tangency*; if $W^s(p)$ and $W^u(p)$ intersect transversally at q we say that q is a point of *transversal homoclinic intersection*. Similarly, homoclinic points associated to a periodic saddle point p, with period k, can be defined by replacing the map F by $G = F^k$ in the definitions above. Notice that transversal homoclinic orbits are structurally stable, that is they are persistent against small perturbations of the map F.

Poincaré already observed that a homoclinic point implies a complicated geometric structure of both the stable and unstable manifolds, as illustrated in Figure 3a. When q is a transversal homoclinic point, then each point $F^n(q)$, $n \in \mathbb{Z}$, is also a transversal homoclinic point. Consequently, both the unstable manifold and the stable manifold oscillate wildly, accumulating onto themselves



(a) The unstable manifold accumulates onto itself.



(b) F^N has a full horseshoe in the rectangular region R, i.e. F^N maps R (as indicated in (a)) onto a horseshoe-like figure $F^N(R)$ folded over R.

FIGURE 3.—Homoclinic tangles and horsehoes.

and so-called *homoclinic tangles* arise. These homoclinic tangles already indicate the complexity of the dynamics and some form of sensitivity to initial states. Furthermore, Smale (1965) showed that a homoclinic point implies existence of (infinitely many) horseshoes. More precisely, for any integer $N > 0$ sufficiently large, there is a rectangular region R, such that the image $F^N(R)$ has the form of a horseshoe folded over the region R as indicated in Figure 3b. We say that the map $G = F^N$ has a (*full*) *horseshoe*. Smale proved that this geometric configuration implies that the map G has an invariant Cantor set with infinitely many periodic points, an uncountable set of chaotic orbits and G has sensitive

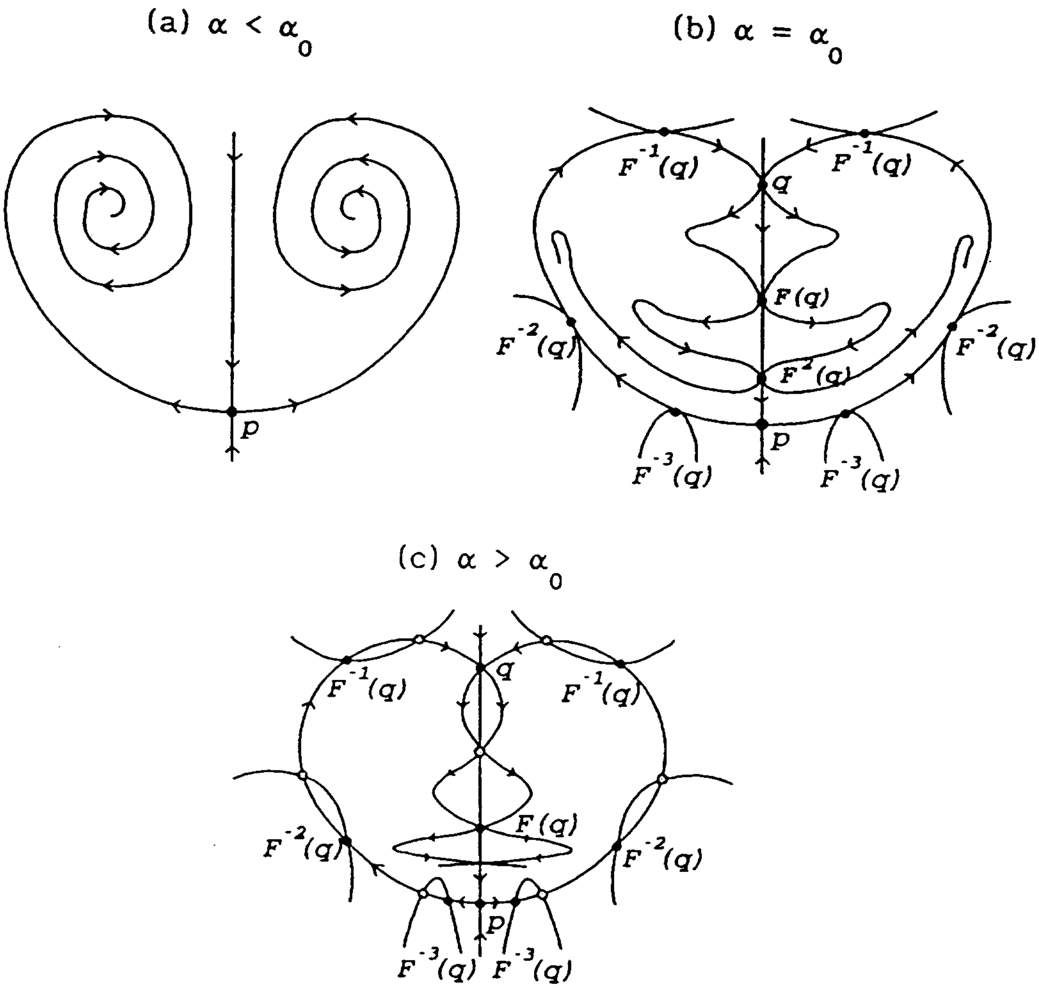


FIGURE 4.—Homoclinic bifurcation in a one-parameter family F_α of non-invertible maps for $\alpha = \alpha_0$. The map F_α is symmetric with respect to the line $x = 0$ and the eigenvalues λ and μ of $JF_\alpha(p)$ satisfy $\mu < -1 < 0 < \lambda < 1$.

dependence on initial states in the invariant Cantor set. Notice, however, that a horseshoe is not an attractor of the dynamical system. This situation is usually called *topological chaos*.[9]

Now consider a one-parameter family of maps F_α with a saddle fixed point p_α (depending upon the parameter α). We say that F_α exhibits a *homoclinic bifurcation*, associated to the saddle point p_α, at $\alpha = \alpha_0$ if (see Figure 4):

(i) for $\alpha < \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have no intersection point $q \neq p_\alpha$;

(ii) for $\alpha = \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have a point of homoclinic tangency;

(iii) for $\alpha > \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have a transversal homoclinic intersection point $q \neq p_\alpha$.

[9] For a discussion of the notion of topological chaos, see e.g. Grandmont (1985, pp. 1026–1027) or Hommes (1991, pp. 11–12).

Similarly, homoclinic bifurcations for *periodic* saddle points can be defined. Recently it has been shown that for two-dimensional nonlinear discrete systems there is a close relation between homoclinic bifurcations and existence of strange attractors for a large set of parameter values. More precisely the following theorem holds:[10]

THEOREM ("Strange Attractor Theorem"; Benedicks and Carleson (1991) and Mora and Viana (1993); see also Palis and Takens (1993, pp. 138–148)): *In generic one-parameter families exhibiting a homoclinic bifurcation at* $\alpha = \alpha_0$, *under the additional assumption that the period k saddle point* p_α *of* F_α *is dissipative* (*i.e. for* $\alpha = \alpha_0$ *the product of the eigenvalues* λ *and* μ *of* $JF^k(p_\alpha)$ *satisfies* $|\lambda\mu| < 1$), *there exists a set of* α*-values in the interval* $(\alpha_0, \alpha_0 + \varepsilon)$ *of positive Lebesgue measure, for which the map* F_α *has a strange attractor.*

This result implies chaotic long run dynamical behavior for a large set of parameters, in generic systems exhibiting homoclinic bifurcations and explains why "strange attractors" are observed in numerical experiments.

3.4. *Global Dynamics*

Now let us return to the cobweb model with rational versus naive expectations. It turns out that as the intensity of choice to switch predictors becomes high, complicated global A.R.E. dynamics arises due to homoclinic bifurcations associated to dissipative periodic saddle points. The geometric shape of the unstable manifold of the steady state plays a key role in understanding the *economic mechanism* responsible for these erratic price fluctuations.

The stable manifold of the steady state E contains the vertical segment $S = \{(p,m) \mid p = 0, -1 \leq m \leq 1\}$, since every point in S is mapped onto E. Figure 5 shows pictures of the unstable manifold of the steady state, for increasing values of the intensity of choice β.[11] Recall from Theorem 3.1 that the steady state has an eigenvalue $\lambda_1 = 0$ and another eigenvalue $\lambda_2 < -1$ for β large. The unstable manifold therefore has two branches. For $\beta = 2$ (Figure 5a) the two branches spiral towards the two points of the stable two-cycle and for $\beta = 3$ (Figure 5b) the branches of the unstable manifold of the steady state

[10] For a definition of strange attractor, see footnote 8. The generic conditions in the theorem are informally denoted as C^3-linearizability, first order (parabolic) tangency, and positive speed (see Palis and Takens (1993, pp. 35–36) and Takens (1992, pp. 192–193)). For real analytic maps these generic conditions have been considerably weakened by Takens (1992), who showed that the following conditions are sufficient: (i) the map is real analytic, (ii) $f(\alpha) = -\ln(\mu(\alpha))/\ln(\lambda(\alpha))$, where $\mu(\alpha)$ and $\lambda(\alpha)$ are the eigenvalues of $JF^k(p_\alpha)$, is not constant, and (iii) there is an "inevitable tangency" (Takens (1992, pp. 192–193)). In the cobweb model with rational versus naive expectations and linear demand and supply, Takens' conditions are satisfied.

[11] Pictures in Figure 5 have been obtained by plotting 20 iterates for 1000 equally spaced points on a small unstable eigenvector of the Jacobian matrix at the steady state. According to the λ-lemma (Guckenheimer and Holmes (1983, pp. 247–248)) in this way finite segments of the unstable manifold can be approximated.

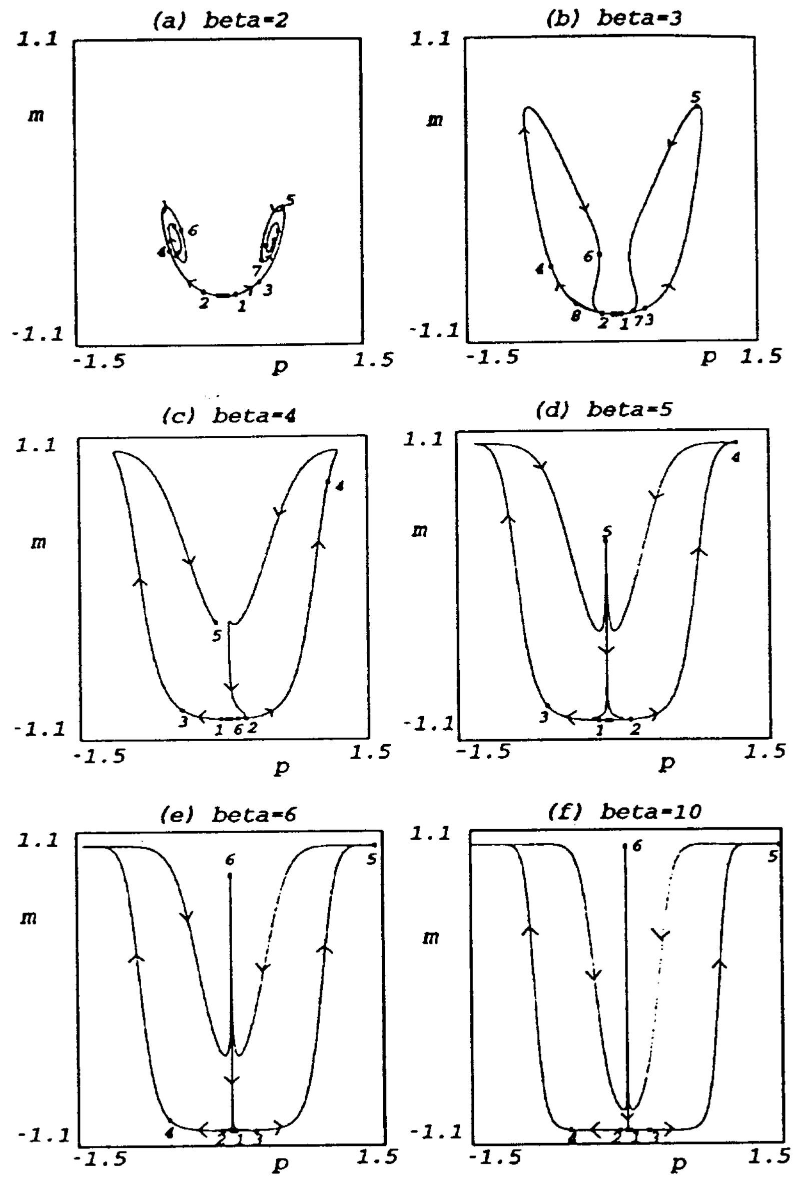


FIGURE 5.—Unstable manifolds of the steady state E for increasing values of the intensity of choice β. For high values of β the system is close to having a homoclinic orbit. The numbers in the figures indicate the dynamics.

spiral around the two points of the unstable two-cycle. As β increases the two branches move closer to each other, approaching the segment S in the stable manifold. In fact, for $\beta \geq 5$ (Figure 5d–f) part of the unstable manifold can hardly be distinguished from the vertical stable segment S. The pictures therefore suggest that for large values of β the system is close to having a homoclinic orbit.

Our strategy to prove that indeed strange attractors arise for high values of the intensity of choice, will be as follows. First we investigate the unstable manifold $W^u_\infty(E)$ of the steady state for the limiting case $\beta = +\infty$. Next we derive an "almost homoclinic tangency lemma," stating that the unstable manifold $W_\beta^u(E)$ and the stable manifold $W_\beta^s(E)$ become "almost tangent" as β tends to infinity. Using the geometric shape of $W_\beta^u(E)$ we then derive two lemmas concerning the creation of horseshoes and homoclinic bifurcations associated to dissipative period $2N$ saddles, as β becomes large.[12] Finally, by applying the "strange attractor theorem" discussed above, we conclude that the A.R.E.D. is chaotic for a positive Lebesgue measure set of (high) β-values.

The Limiting Case $\beta = +\infty$

In order to understand the global geometric shape of the unstable manifold of the steady state for high but finite values of β, it will be useful to consider the case $\beta = +\infty$ first, with (3.6) or (3.9) replaced by

$$m_{t+1} = \begin{cases} +1, & \text{if } \frac{b}{2}(p_{t+1} - p_t)^2 > C, \\ -1, & \text{if } \frac{b}{2}(p_{t+1} - p_t)^2 \le C, \end{cases} \tag{3.12}$$

so agents switch infinitely fast between rational and naive expectations. The limiting case $\beta = +\infty$ is important from an economic viewpoint as well, since for $\beta = +\infty$ in each period *all* agents choose the optimal predictor. We have the following result.

THEOREM 3.2: *For $\beta = +\infty$, even when the market is locally unstable (i.e. $b/B > 1$) and when information costs $C > 0$, the system always converges to the saddle point equilibrium steady state $E = (0, -1)$.*

Hence, for $\beta = +\infty$ the steady state equilibrium $E = (0, -1)$ attracts all initial states in finite time. Notice however that the steady state E is a *saddle point* and therefore it is *locally unstable*. Adding a small amount of noise to the system would lead to very irregular price fluctuations, which in fact are very similar to the fluctuations with β large but $\beta < +\infty$, considered below.

It will be useful to investigate the unstable manifold of the steady state in the limiting case $\beta = +\infty$. Define $C' = (B/(B+b))\sqrt{(2C/b)}$ and the point $A_0 = (C', -1)$. The reader can check that A_0 is exactly the point where all agents are naive (i.e., $m_t = -1$), while net realized profits for rational and naive agents will be equal, or equivalently $(b/2)(p_{t+1} - p_t)^2 = C$. Hence, from (3.12) it follows that A_0 is exactly the point where all agents switch from being naive to become rational. The segment EA_0 lies in the unstable manifold of the steady state E.

[12] At the end of Subsection 3.5 we will use the DUNRO computer program to detect what seems to be the primary heteroclinic and homoclinic bifurcations as β increases.